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LETTER TO THE EDITOR

Storage capacity for hierarchically correlated patterns

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Abstract. The storage capacity of a Hopfield model with patterns forming a two-level hierarchy is calculated without reference to a special learning rule. If the number of different classes becomes very large strong correlations within the classes *decrease* the storage capacity in striking contrast to what is known about one-level hierarchies, i.e. patterns with magnetisation.

Large networks of N two-state neurons can function as associative memories (Hopfield 1982) and have raised a lot of interest in the statistical physics community (van Hemmen and Morgenstern 1987). In particular it has been shown for different statistics of the patterns and several learning rules that one can store at most $p = O(N)$ patterns for $N \to \infty$ (Amit *et al* 1985, 1987, Kanter and Sompolinsky 1987, van Hemmen 1987, Gardner 1988), which is rather modest since these systems are known to possess $O(\exp(aN))$ metastable states (Gardner 1986). One of the most exciting problems in this context concerns hierarchies of patterns. By this we have in mind the properties of a homogeneous network storing hierarchically correlated patterns (Parga and Virasoro 1986, Toulouse *et al* 1986, Feigelman and Ioffe 1987, Cortes *et al* 1987, Bös *et al* 1988), a situation rather different from that of hierarchically structured networks (Dotsenko 1985, 1986, Gutfreund 1988, Sourlas 1988). Hierarchies of patterns are interesting on the one hand because the brain is known to classify the data to be stored (Simon 1962, Parga and Virasoro 1986, Virasoro 1988) and on the other hand since an ultrametric organisation of the low-energy states occurs spontaneously in randomly connected networks (Mézard *et al* 1984). So far the storage capacity for hierarchically correlated patterns has been discussed using special learning rules only (Parga and Virasoro 1986, Feigelman and Ioffe 1987, Cortes *et al* 1987, Bös *et al* 1988). The results are again $p = O(N)$; however, it remains unclear whether this is due to the statistics of the patterns or the learning rule implemented.

In this letter we determine the critical storage capacity for correlated random patterns forming a two-level hierarchy without reference to a special learning rule. This is done by calculating the partial volume $\langle\langle V \rangle\rangle$ in the phase space of interactions which contains those combinations of couplings J_{ij} that stabilise all the patterns. This rather powerful approach was introduced recently by Elizabeth Gardner (Gardner 1988). The application of this method to hierarchically correlated patterns has also been suggested by Virasoro (1988), who considered a special type of two-level hierarchy. Here we are concerned with the most general form of a regular hierarchy with two levels; in particular we consider the case of infinitely many classes containing a finite number of patterns. The patterns to be stored are defined by

$$\xi_i^{\mu} = \xi_i^{(1),\mu'} \xi_i^{(0),\mu} \qquad (1a)$$

where the $\xi_i^{(r),\mu}$ are independent random variables with distribution

$$P(\xi_i^{(r),\mu}) = \frac{1+m_r}{2}\,\delta(\xi_i^{(r),\mu}-1)\frac{1-m_r}{2}\,\delta(\xi_i^{(r),\mu}+1) \tag{1b}$$

and $\mu' = [\mu/z_0]$ denotes the largest integer smaller than μ/z_0. z_0 and z_1 are the branching ratios at the zeroth and the first level respectively, hence $\mu' = 1, \ldots, z_1$, $\mu = 1, \ldots, z_0 z_1$.

The central quantity to be calculated is the nth power of the partial volume V averaged over the statistics (1) of the patterns in the limits $N\to\infty$, $n\to 0$. The starting expression is the same as in Gardner's case (Gardner 1988):

$$\langle\!\langle V^n\rangle\!\rangle = \int \prod_{\alpha=1}^{n}\prod_{j\neq i}^{N} \mathrm{d}J_{ij}^\alpha \prod_{\alpha=1}^{n}\delta\Big(\sum_{j\neq i}(J_{ij}^\alpha)^2 - N\Big)\Big\langle\!\!\Big\langle \prod_{\mu=1}^{p}\prod_{\alpha=1}^{n}\theta\Big(\xi_i^\mu\frac{1}{N^{1/2}}\sum_{j\neq i}J_{ij}^\alpha\xi_j^\mu - \kappa\Big)\Big\rangle\!\!\Big\rangle$$
$$\times\Big[\int\prod_{\alpha=1}^{n}\prod_{j=:i}\mathrm{d}J_{ij}^\alpha\delta\Big(\sum_{j\neq i}(J_{ij}^\alpha)^2-N\Big)\Big]^{-1}. \tag{2}$$

The δ functions make $J_{ij}^\alpha = \mathrm{O}(1)$ whereas the product of the θ-functions ensures the stability of the patterns to be stored. The disorder-independent part can be handled completely in the same way as for uncorrelated patterns (Gardner 1988). Introducing integral representations of the θ-functions we find for the remaining part

$$\Big\langle\!\!\Big\langle\int_\kappa^\infty\prod_{\mu,\alpha}\frac{\mathrm{d}\lambda_\mu^\alpha}{2\pi}\int\prod_{\mu,\alpha}\mathrm{d}x_\mu^\alpha\exp\Big\{\mathrm{i}\sum_{\mu,\alpha}x_\mu^\alpha\lambda_\mu^\alpha - \frac{\mathrm{i}}{N^{1/2}}\sum_{\mu,\alpha}x_\mu^\alpha\xi_i^\mu\sum_{j\neq i}J_{ij}^\alpha\xi_j^\mu\Big\}\Big\rangle\!\!\Big\rangle. \tag{3}$$

Using (1) we first average over the $\xi_j^{(0),\mu}$ and after introducing the variables $q^{\alpha\beta}$ and $\Lambda_{\mu'}^\alpha$ defined by the δ-functions below we get to leading order in N

$$\int\prod_{\mu',\alpha}\mathrm{d}\Lambda_{\mu'}^\alpha\int\prod_{\alpha<\beta}\mathrm{d}q^{\alpha\beta}\prod_{\alpha<\beta}\delta\Big(q^{\alpha\beta}-\frac{1}{N}\prod_{j\neq i}J_{ij}^\alpha J_{ij}^\beta\Big)$$
$$\times\Big\langle\!\!\Big\langle\prod_{\mu',\alpha}\delta\Big(\Lambda_{\mu'}^\alpha - \frac{1}{N^{1/2}}\xi_i^{(1),\mu'}\sum_{j\neq i}J_{ij}^\alpha\xi_j^{(1),\mu'}\Big)\Big\rangle\!\!\Big\rangle_{\xi_i^{(1),\mu'},\xi_j^{(1),\mu'}}$$
$$\times\prod_{\mu'}\exp\{z_0 G^{(0)}(\bar\Lambda_{\mu'},\bar{\bar q})\} \tag{4}$$

where

$$G^{(0)}(\bar\Lambda_{\mu'},\bar{\bar q}) = \ln\Big\langle\!\!\Big\langle\int_\kappa^\infty\prod_\alpha\frac{\mathrm{d}\lambda^\alpha}{2\pi}\int\mathrm{d}x^\alpha\exp\Big\{\mathrm{i}\sum_\alpha x^\alpha\lambda^\alpha - \mathrm{i}m_0\xi_i^{(0)}\sum_\alpha x^\alpha\Lambda_{\mu'}^\alpha$$
$$-\frac{1-m_0^2}{2}\sum_\alpha(x^\alpha)^2 - \frac{1-m_0^2}{2}\sum_{(\alpha,\beta)}x^\alpha x^\beta q^{\alpha\beta}\Big\}\Big\rangle\!\!\Big\rangle_{\xi_i^{(0)}}. \tag{5}$$

Using an integral representation for the second δ-function in (4) we find

$$\int\prod_{\alpha<\beta}\mathrm{d}q^{\alpha\beta}\prod_{\alpha<\beta}\delta\Big(q^{\alpha\beta}-\frac{1}{N}\sum_{j\neq i}J_{ij}^\alpha J_{ij}^\beta\Big)\Big\langle\!\!\Big\langle\int\prod_{\mu',\alpha}\frac{\mathrm{d}\Lambda_{\mu'}^\alpha}{2\pi}\int\prod_{\mu',\alpha}\mathrm{d}X_{\mu'}^\alpha$$
$$\times\exp\Big\{\mathrm{i}\sum_{\mu',\alpha}X_{\mu'}^\alpha\Lambda_{\mu'}^\alpha - \frac{1}{N^{1/2}}\sum_{\mu',\alpha}\xi_i^{(1),\mu'}X_{\mu'}^\alpha\textstyle\sum_j J_{ij}^\alpha\xi_j^{(1),\mu'}$$
$$+z_0\sum_{\mu'}G^{(0)}(\bar\Lambda_{\mu'},\bar{\bar q})\Big\}\Big\rangle\!\!\Big\rangle. \tag{6}$$

Comparing (6) with (3) one realises that by averaging over the lowest level auxiliary variables $\xi_i^{(0),\mu}$ one can map the determination of $\langle\langle V^n\rangle\rangle$ for a hierarchy of two levels with $p = z_0 z_1$ patterns on that of $\langle\langle V'^n\rangle\rangle$ for a hierarchy of one level with $p' = z_1$ patterns. This renormalisation group procedure becomes particularly valuable in the analysis of pattern hierarchies with a number of levels tending to infinity for $N \to \infty$ (Engel 1990).

The average over the $\xi_j^{(1),\mu'}$ in (6) can now be performed completely analogously to that over the $\xi_j^{(0),\mu}$ and after standard manipulation (Gardner 1988) we find for the numerator of (2)

$$\int \prod_{\alpha<\beta} \mathrm{d}q^{\alpha\beta} \frac{\mathrm{d}F^{\alpha\beta}}{2\pi/N} \int \prod_\alpha \mathrm{d}M^\alpha \frac{\mathrm{d}E^\alpha}{4\pi} \frac{\mathrm{d}K^\alpha}{2\pi/N^{1/2}}$$
$$\times \exp\left\{ N\left[-\sum_{\alpha<\beta} F^{\alpha\beta} q^{\alpha\beta} + \tfrac{1}{2}\sum_\alpha E^\alpha + \frac{1}{N^{1/2}} \sum_\alpha M^\alpha K^\alpha \right.\right.$$
$$\left.\left. + g(\bar{\bar{F}}, \bar{E}, \bar{K}) + \frac{z_1}{N} G^{(1)}(\bar{M}, \bar{\bar{q}}) \right]\right\} \tag{7}$$

where

$$g(\bar{\bar{F}}, \bar{E}, \bar{K}) = \ln\left[\int \prod_\alpha \mathrm{d}J^\alpha \exp\left\{ \sum_{\alpha<\beta} F^{\alpha\beta} J^\alpha J^\beta - \tfrac{1}{2}\sum_\alpha E^\alpha (J^\alpha)^2 - \sum_\alpha \kappa^\alpha J^\alpha \right\}\right] \tag{8}$$

and

$$G^{(1)}(\bar{M}, \bar{\bar{q}}) = \ln\left[\left\langle\!\!\left\langle \int \prod_\alpha \frac{\mathrm{d}\Lambda^\alpha}{2\pi} \int \prod_\alpha \mathrm{d}X^\alpha \exp\left\{ \mathrm{i}\sum_\alpha X^\alpha \Lambda^\alpha - \mathrm{i} m_1 \xi_i^{(1)} \sum_\alpha X^\alpha M^\alpha \right.\right.\right.\right.$$
$$\left.\left.\left.\left. - \frac{1-m_1^2}{2}\sum_\alpha (X^\alpha)^2 - \frac{1-m_1^2}{2}\sum_\alpha X^\alpha X^\beta q^{\alpha\beta} + z_0 G^{(0)}(\bar{\Lambda}, \bar{\bar{q}}) \right\}\right\rangle\!\!\right\rangle_{\xi_i^{(1)}} \right]. \tag{9}$$

We now distinguish two cases: the specialist with $z_0 = \mathrm{O}(N)$ and $z_1 = \mathrm{O}(1)$ and the universalist $z_0 = \mathrm{O}(1)$ and $z_1 = \mathrm{O}(N)$. In the first case we find from (9) $G^{(1)}(\bar{M}, \bar{\bar{q}}) = z_0 G^{(0)}(\bar{\Lambda}^{(\mathrm{SP})}, \bar{\bar{q}})$ by a saddle-point argument. Hence in both cases the prefactor of $G^{(1)}$ in (7) is O(1) and the integrals can be calculated using the saddle-point approximation.

We assume the saddle point to be replica symmetric which is sensible since the volume V is connected. The critical storage capacity α_c is then given by the solution of the saddle-point equations for $q = q^{\alpha\beta}$, $\alpha \neq \beta$, and $M = M^\alpha$ in the limit $q \to 1$ (Gardner 1988). For $z_0 = \mathrm{O}(1)$ and $z_1 = \mathrm{O}(N)$ we obtain in this way (Engel 1990)

$$\frac{1}{\alpha_c} = \left\langle\!\!\left\langle \int \mathrm{D}t_1 \int_{-\kappa''}^{\infty} \mathrm{D}t_0 (\kappa'' + t_0)^2 \right\rangle\!\!\right\rangle_{\xi_i^{(0)}, \xi_i^{(1)}} \tag{10}$$

where

$$\kappa'' = (1 - m_0^2)^{-1/2} [\kappa - m_0 \xi_i^{(0)} m_1 \xi_i^{(1)} M + m_0 \xi_i^{(0)} (1 - m_1^2)^{1/2} t_1] \tag{11}$$

and M has to be determined from

$$0 = \left\langle\!\!\left\langle \int \mathrm{D}t_1 \int_{-\kappa''}^{\infty} \mathrm{D}t_0 (\kappa'' + t_0) \xi_i^{(0)} \xi_i^{(1)} \right\rangle\!\!\right\rangle_{\xi_i^{(1)}, \xi_i^{(1)}}. \tag{12}$$

Here $\int \mathrm{D}t \ldots$ means $\int [\mathrm{d}t/(2\pi)^{1/2}]\, \mathrm{e}^{-t^2/2} \ldots$.

If $z_0 = O(N)$ and $z_1 = O(1)$ the saddle point equation for M is to be replaced by one for Λ (cf (9)) and we find

$$\frac{1}{\alpha_c} = \left\langle\!\left\langle \int_{-\kappa'}^{\infty} Dt_0(\kappa' + t_0)^2 \right\rangle\!\right\rangle_{\xi_i^{(0)}} \tag{13}$$

where

$$\kappa' = (1 - m_0^2)^{-1/2}[\kappa - m_0 \xi_i^{(0)} \Lambda] \tag{14}$$

and Λ is given by

$$0 = \left\langle\!\left\langle \int_{-\kappa'}^{\infty} Dt_0(\kappa' + t_0)\xi_i^{(0)} \right\rangle\!\right\rangle_{\xi_i^{(0)}}. \tag{15}$$

Equations (13)-(15) are exactly those found by Gardner (1988) for patterns with magnetisation m_0. This means that if one can store p patterns with magnetisation m_0 one can also store $z_1 = O(1)$ classes of patterns all containing p/z_1 patterns with magnetisation m_0 irrespective of the correlations between the classes. In particular one finds $\alpha_c \to \infty$ for $m_0 \to 1$. This result has also been reported by Virasoro (1988) for the special case $m_1 = 0$.

The more interesting case is the universalist one described by (10)-(12). Performing the remaining averages and Gaussian integrals one finds after some algebra

$$\alpha_c(m_0, m_1, \kappa) = \frac{1 - m_0^2}{1 - m_0^2 m_1^2} \alpha_c^G(m_0 m_1, \kappa) \tag{16}$$

where $\alpha_c^G(m, \kappa)$ is the critical storage capacity for patterns with magnetisation m and stability κ as calculated by Gardner (1988). Note that $m_0 m_1$ is the overlap between patterns belonging to different classes. Equation (16) shows $\alpha_c \to 0$ if $m_0 \to 1$ in striking difference to the specialist case described by (13)-(15). If one wants to store infinitely many classes of patterns then strong correlations within each class make a perfect stabilisation of all patterns very difficult. In figure 1 α_c is plotted as a function of m_0 for different values of m_1. Note that $\alpha_c(m_0 = 1, m_1, \kappa)$ must be equal to $z_0 \alpha_c^G(m_1, \kappa) > 0$ since for $m = 1$ one has to store simply z_1 patterns with magnetisation m_1. Hence

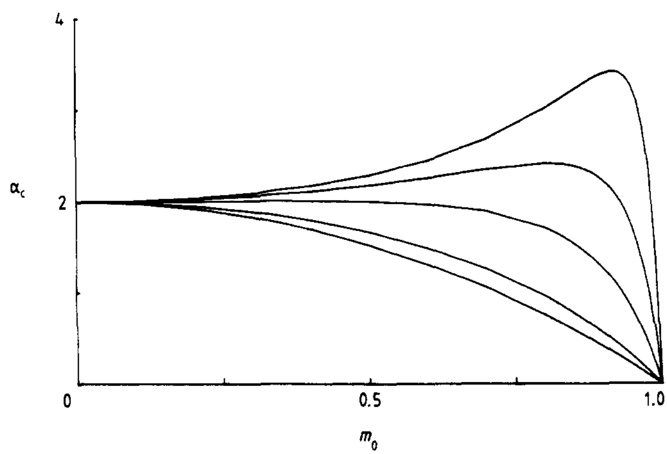

Figure 1. Critical storage capacity α_C as a function of m_0 for $m_1 = 0.2, 0.5, 0.8, 0.9, 0.95$ (from bottom to top) for a hierarchy with infinitely many classes.

$\alpha_c(m_0, m_1, \kappa)$ must be discontinuous at $m_0 = 1$. This seems less strange if one keeps in mind that it is impossible to store perfectly two patterns which just differ by one bit. So patterns which are *almost* identical are the main problem. It is surprising that these complications do not occur for hierarchies with finitely many classes.

In conclusion we have shown that the organisation of patterns in a regular two-level hierarchy does not improve the critical storage capacity in comparison with a 'one-level' hierarchy, i.e. patterns with magnetisation. Note that due to the additional correlations the information content of a two-level hierarchy with m_0 and m_1 is less than that of patterns with magnetisation $m_0 m_1$. Moreover since (16) is valid for all κ one should not expect an improvement of the basins of attraction by the hierarchial organisation. It might be that by tolerating a small error in the retrieval (Gardner and Derrida 1988) one can improve α_c significantly as in the case of the Hopfield model (Amit *et al* 1985) although this seems not very likely. Besides the study of hierarchies with infinitely many levels it is therefore most promising to consider irregular hierarchies of the type found in spin glasses (Mézard and Virasoro 1985).

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References

Amit D J, Gutfreund H and Sompolinsky H 1985 *Phys. Rev. Lett.* **55** 1530
—— 1987 *Phys. Rev.* A **35** 2293
Bös S, Kühn R and van Hemmen J L 1988 *Z. Phys.* B **71** 261
Cortes C, Krogh A and Hertz J A 1987 *J. Phys. A: Math. Gen.* **20** 4448
Dotsenko V S 1985 *J. Phys. C: Solid State Phys.* **18** L1017
—— 1986 *Physica* **140A** 410
Engel A 1990 Storage of hierarchically correlated patterns *J. Phys. A: Math. Gen.* **23** L285
Feigelman M V and Ioffe L B 1987 *Int. J. Mod. Phys.* B **1** 51
Gardner E 1986 *J. Phys. A: Math. Gen.* **19** L1047
—— 1988 *J. Phys. A: Math. Gen.* **21** 257
Gardner E and Derrida B 1988 *J. Phys. A: Math. Gen.* **21** 271
Gutfreund H 1988 *Phys. Rev.* A **37** 570
Hopfield J J 1982 *Proc. Natl Acad. Sci. USA* **79** 2554
Kanter I and Sompolinsky H 1987 *Phys. Rev.* A **35** 380
Mézard M, Parisi G, Sourlas N, Toulouse G and Virasoro M A 1984 *J. Physique* **45** 843
Mézard M and Virasoro M A 1985 *J. Physique* **46** 1293
Parga N and Virasoro MA 1986 *J. Physique* **47** 1857
Simon H 1962 *Proc. Am. Phil. Soc.* **106** 467
Sourlas N 1988 *Europhys. Lett.* **7** 749
Toulouse G, Dehaene S and Changeux J-P 1986 *Proc. Natl Acad. Sci. USA* **83** 1685
van Hemmen J L 1987 *Phys. Rev.* A **36** 1953
van Hemman J L and Morgenstern I (eds) 1987 *Heidelberg Coll. on Glassy Dynamics (Lecture Notes in Physics 275)* (Berlin: Springer)
Virasoro M A 1988 *Europhys. Lett.* **7** 293